Tolerances in Symbolic Network Analysis

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ABSTRACT

Tolerances of electronic circuit parameters play an important role in our judgement of circuit functionality. They must be considered both at the circuit design and testing stages. This paper discusses a problem of tolerances in a symbolic network analysis. Using topological analysis combined with operations on discrete random variables, we can predict behavior of a circuit with toleranced parameters. Automatic simplification of symbolic network functions is used to facilitate operations on discrete variables representing network parameters. Our discussion is complemented with illustrative examples.

INTRODUCTION

Due to imperfections of technological processes, parameters of fabricated circuits differ from the designed values. Consequently, variations of circuit parameters change the expected circuit response. Predicted changes must be compared against the design requirements in order to qualify a design as a working one. Our knowledge of statistical properties of the circuit response can be used to determine economics of circuit fabrication in terms of a production yield or a cost per working unit.

Various tools are used to perform tolerance analysis. We can classify them into two groups. The first group contains deterministic methods like small and large change sensitivity analysis, and worst case analysis. However, these methods are not very accurate and cannot accommodate real probability density functions of the network parameters. The second group contains statistical methods like Monte Carlo analysis, analytical method or moments method. Monte Carlo method is very popular and gives results of good accuracy [1]. Its drawback is very high computational cost caused by repetitive analysis of a circuit with parameters statistically varied. Analytical method, which determines probability density function of the circuit response analytically, is very complex and until now is limited to very simple or special cases [2]. Moments method, which expands circuit transfer function in Taylor series, is relatively fast but not very accurate [3]. A compromise based on the generalized quantile arithmetic was proposed by Glesner *et al.* [4]. This last method gives results comparable to Monte Carlo method in time which is two orders of magnitudes smaller.

In our paper we combine Glesner's approach with the topological analysis [5], where tolerances of all network parameters can be represented. Hierarchical topological analysis can be performed for large circuits in time comparable to numerical analysis [6,7]. Both direct and hierarchical topological analyses may face a problem of a large number of terms in topological formulas. In the following section we discuss how to ease this problem by simplification of symbolic network functions. Then we discuss tolerance analysis by discretization of random variables.

SIMPLIFICATION OF NETWORK FUNCTIONS

Symbolic form of a network function is often preferred over numerical results as it gives a designer better insight to the effect of network parameters on the network function. However, this is hardly true if the number of components in symbolic formula increases. Number of terms can become large in networks having as little as 5–6 nodes. A practical solution to this problem in hand analysis is to neglect some of the less important parameters. However, this approach can only be used by experienced designer, who knows which parameters play dominant role in different frequency intervals. On the other hand, the process of simplification of symbolic network function can be performed automatically using topological analysis. In this section we describe such simplification for the direct topological analysis, however similar approach can be used in a hierarchical topological analysis.

A symbolic network function obtained by topological analysis is a multivariable rational function and both its numerator and denominator can be sorted according to powers of complex variable s:

$$M(s,y) = \sum_i s^i W_i \;, \tag{1}$$

where W_i is a polynomial of network parameters y:

$$W_i = \sum_{t\in T} w_t = \sum_{t\in T} \prod_{e\in t} y_e \,. \tag{2}$$

and a network parameter y_e may be a symbol or a numerical value. Let us consider the following example.

Example

An equivalent model of a simple transistor circuit is shown on Fig.1. Symbolic voltage transfer function of the network is given by

$$T_v(s,y) = \frac{N(s,y)}{D(s,y)} \tag{3}$$

N(s,y) and D(s,y) can be generated using a direct topological analysis program [10]. Let us assume that symbolic parameters have the following nominal values:

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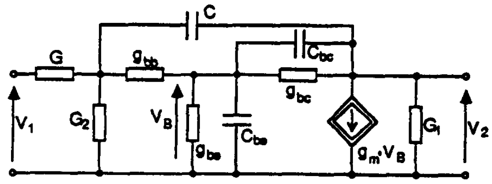

Fig. 1 A simple transistor circuit.

G=0.1 mS, G_1=0.5 mS, C=10 nF, C_{be}=50 pF, C_{bc}=3 pF, g_{bb}=10 mS, g_{be}=0.8 mS, g_{bc}=1 μS,g_m=80 mS,G_2=2.5 μS. Analyzing the symbolic formula, we have observed that components having the same power of complex variable s differ by several orders of magnitude. For example components of denominator, which are multiplied by s, vary from 7.5E–24 for the term $C_{bc}G_2g_{be}$ up to 8E–12 for $C\, g_{bb}g_m$.

In order to simplify further analysis we propose to replace the components satisfying condition

$$w_t < A_i \tag{4}$$

by their nominal values, where the truncating level A_i is determined either from

$$A_i = \epsilon\, W_i \tag{5}$$

or

$$A_i = \epsilon \max(|w_t|) \tag{6}$$

and ϵ ($\epsilon << 1$) depends on the desired accuracy of symbolic expression.

Thus, taking into account numerical and symbolic parameters separately, we obtain components of symbolic formula in the following form:

$$W_i \approx w_0 + \sum_{t \in T_s} w_{t_d} \prod_{e \in t_s} y_e , \tag{7}$$

where w_0 is the sum of all components with no symbolic terms and nominal values satisfying (4), w_{t_d} is the numerical part of a significant component, and t_s is its symbolic part.

In our example if we use (6) with $\epsilon = 0.001$ then the simplified transfer function will be described by

$N'(s,y) = s^2\, C\, G\, (\, C_{be} + C_{bc}) + s\, [\, G\, C\, (\, g_{bb} + g_{be}) + 4E{-}18\,) - G\, g_{bb}\, g_m$

and

$D'(s,y) = s^3\, C\, C_{be}C_{bc} + s^2\{\, C\, [\, C_{bc}(\, g_{be} + g_m + G_1) + C_{be}(\, G + G_1 + g_{bb})] + 5.000375E{-}24\, \} + s\, \{C\, [g_{bb}(\, G + g_m + g_{be}) + g_m G_1] + 1.83341363825E{-}14\} + G\, [G_1(\, g_{bb} + g_{be}) + g_{bc}g_m] + g_{bb}[\, g_{bc}(\, g_{be} + g_m + G_1) + G_1(\, G_2 + g_{be})] + 2.358225E{-}12$

In this simple example number of symbolic terms in numerator and denominator was reduced from 62 to 24, however much larger reductions exceeding 99% are observed in larger networks, particularly when circuit parameters vary several orders of magnitude.

The proposed algorithm of automatic simplification of symbolic network functions has the following properties:

- a symbolic function obtained is valid for all frequencies, which is an advantage over simplifications introduced by a designer, usually valid in a limited frequency range [8],
- symbolic parameters are present only in those parts of a formula where they are significant, again this is in contrast to designer simplification where some parameters will be removed from the circuit model as nonsignificant,
- formula is exact if network parameters assume their nominal values,
- by appropriate selection of ϵ, accuracy of a symbolic function may be kept as high as required.

Automatic simplification of network functions neither accelerate topological analysis nor allow to increase the size of a circuit analyzed. This problem has to be resolved by using effective topological analysis techniques like these presented in [6,7]. However, automatic simplification facilitates further work in applications, where the knowledge of symbolic functions is essential. One such an application is tolerance analysis by discretization of random variables, which is discussed in the next section.

TOLERANCE ANALYSIS

In this section we combine tolerance analysis by discretization of random variables introduced in [4] with topological analysis. We discuss properties of such approach and present results of computer simulation.

Operations on Discrete Random Variables

Let $\tilde{X}$ be a continuous random variable with probability density function $v(\tilde{X})$. Variable $\tilde{X}$ will be approximated by N–point discrete random variable X [4]

$$X = [\, X_1\, , X_2\, , \dots , X_N\,] \tag{8}$$

with probability density function

$$p(X) = [\, p(X_1), p(X_2), \dots , p(X_N)] \tag{9}$$

Let us assume that circuit parameters $y_1, y_2, \dots, y_n$ are described by N–point discrete random variables $Y_1, Y_2, \dots, Y_n$. Assume also that we know symbolic form of a network function

$$f = f(y_1, y_2, \dots , y_n, \omega\,) \tag{10}$$

Probability density function of the random variable $\tilde{F} = f(\tilde{Y}_1, \tilde{Y}_2, \dots , \tilde{Y}_n, \omega\,)$ will be approximated by a discrete random variable $F = f(Y_1, Y_2, \dots , Y_n, \omega\,)$.

Evaluation of F is much easier than evaluation of $\tilde{F}$ due to operations on discrete random variables. Basic operations needed to evaluate function f are: addition(+), subtraction(−), multiplication(*), and division(/). Symbol (o) is used to represent any one of these operations.

Let U, V, and Z be discrete random variables, and let

$$Z = U \circ V \tag{11}$$

Probability of combination

$$Z_{ij} = U_i \circ V_j \tag{12}$$

is described by 2–dimensional discrete probability density function

$$\pi_{ij} = (1 - |\rho_{UV}|)\, p_U(U_i)\, p_V(V_j) + |\rho_{UV}|\, \pi_1(U_i,V_j), \quad i,j=1,2, \dots ,N. \tag{13}$$

where ρ_{UV} is the correlation coefficient of variables U and V, and

$$\pi_1(U_i,V_j) = \begin{cases} 0 & \text{for } i \neq j \\ p_U(U_i) & \text{for } i=j \end{cases} \quad \rho_{UV} > 0; \qquad \begin{cases} 0 & \text{for } i+j \neq N+1 \\ p_V(V_j) & \text{for } i+j = N+1 \end{cases} \quad \rho_{UV} < 0 \tag{14}$$

Having all values Z_{ij} and corresponding probabilities π_{ij}, pairs (Z_{ij}, π_{ij}) are ordered in increasing values of Z_{ij} and resulting N^2–point discrete variable is reduced to an N–point discrete variable

$$Z = [Z_1, Z_2, \dots ,Z_n] \tag{15}$$

with probability density function

$$p(Z) = [p(Z_1), p(Z_2), \dots ,p(Z_N)] \tag{16}$$

While the correlation coefficients of circuit parameters can be determined from their probability density functions (assumed to be known), further analysis requires determination of covariances and correlation coefficients of discrete variables Z_ν described by (15). Correlation coefficient of two variables Z_ν and Z_μ can be calculated from

$$\rho_{Z_\nu,Z_\mu} = \frac{\mathrm{cov}(Z_\nu,Z_\mu)}{\sigma_{Z_\nu}\, \sigma_{Z_\mu}} \tag{17}$$

where covariance of Z_ν and Z_μ is

$$\mathrm{cov}(Z_\nu, Z_\mu) = \sum_{i=1}^{n} \frac{\mathrm{cov}(Z_\nu,Y_i)\ \mathrm{cov}(Z_\mu,Y_i)}{\sigma^2_{Y_i}} \tag{18}$$

and

$$\mathrm{cov}(Z_\lambda, Y_i) = \tag{19}$$

$$\begin{array}{ll} \mathrm{cov}(Z_\nu,Y_i) + \mathrm{cov}(Z_\mu,Y_i) & \text{for } Z_\lambda=Z_\nu+Z_\mu \\ \mathrm{cov}(Z_\nu,Y_i) - \mathrm{cov}(Z_\mu,Y_i) & \text{for } Z_\lambda=Z_\nu-Z_\mu \\ \bar{Z}_\mu\, \mathrm{cov}(Z_\nu,Y_i) + \bar{Z}_\nu\, \mathrm{cov}(Z_\mu,Y_i) & \text{for } Z_\lambda=Z_\nu * Z_\mu \\ \bar{Z}_\mu\, \mathrm{cov}(Z_\nu,Y_i) - \bar{Z}_\nu\, \mathrm{cov}(Z_\mu,Y_i)/\, \bar{Z}^2_\mu & \text{for } Z_\lambda=Z_\nu/Z_\mu \end{array}$$

where $\bar{Z}_\mu$ is the expected (nominal) value of a variable Z_μ. Standard deviations needed in (17) can be obtained from

$$\sigma^2_{Z_\nu} = \mathrm{cov}(Z_\nu,Z_\nu) \tag{20}$$

If the tolerance analysis is performed for specific frequencies, then the network functions are described by complex variables. In this case we may use complex random variables with operations performed according to the rules for complex variables [9].

Topological Approach

In what follows we discuss tolerance analysis based on direct topological approach, in which a circuit is represented by its unistor graph. Such approach is limited to analysis of small circuits only, but the results are applicable to hierarchical analysis suitable for large circuits [6,7].

At this point let us assume that topological analysis has been performed with automatic simplification of network functions, yielding a transfer function described by the ratio of polynomials M(s,y) (1). Polynomials W_i in (1) are described by (7). If we substitute parameters y_e by their discrete random variables Y_e then we will obtain formulas for the random variables describing terms in numerator and denominator as well as the transfer function. Let discrete variable U_i represents W_i

$$U_i = w_0 + \sum_{t \in T_s} w_{t_d} \prod_{e \in t_s} Y_e . \tag{21}$$

It may happen that the same variable appears more than once in t_s (21). This may be caused by a tree branches having the same weight associated with graphs of some active elements. To indicate possible repetitions of variables, (21) is replaced by

$$U_i = w_0 + \sum_{t \in T_s} w_{t_d} \prod_{e \in t_r} Y_e^{q_e} . \tag{22}$$

where t_r is a subset of t_s representing distinct random variables Y_e with $q_e \geq 1$.

Covariance of U_i and Y_e can be calculated from

$$\mathrm{cov}(U_i,Y_e) = \tag{23}$$

$$\begin{cases} 0 & \text{if } \quad e \notin t_r \qquad \text{else} \\ w_{t_d} \{ \prod_{j \neq e} [\, Y_j^{q_j} + \frac{q_j}{2} (q_j - 1)\, Y_j^{q_j - 2}\, \sigma^2_{Y_j}] \}\, q_e\, Y_e^{q_e - 1}\, \sigma^2_{Y_e} & \end{cases}$$

and standard deviation

$$\sigma^2_{U_i} = w^2_{t_d} \sum_{i=1}^{n} [\, \sigma^2_{Y_i} (\, q_i\, Y_i^{q_i - 1} \prod_{\substack{e \in t_r \\ e \neq i}} Y_e^{q_e})^2] \tag{24}$$

After all U_i are found (both for the terms in numerator and denominator) the network function will be obtained as a rational function of complex variable s with coefficients represented by discrete random variables. Number of operations needed to obtain all coefficients is equal to

$$N_u = \text{card}(T)\ (v - k) \tag{25}$$

where card(T) is the cardinality of the set of k-trees T, v is the number of graph vertices. N_u is large even for small networks and in direct topological analysis the size of analyzed networks is limited to 10–15 nodes. Using topological analysis with automatic simplification of the symbolic network functions reduces computational efforts with slight decrease in accuracy of the results.

Results of Computer Simulation

Fig. 2 shows an active filter circuit analyzed by program DISTOR4 [10]. DISTOR4 implements tolerance analysis with automatic simplification of symbolic network functions obtained by direct topological analysis of a circuit represented by its unistor graph. Nominal values assumed in tolerance analysis are as follows: $R_a=R_b=R_c=R_f=10k\Omega$, $R_e=1M\Omega$, $C_a=C_b=2nF$. All circuit parameters have normal distribution with tolerances $t = 3\sigma = \pm 2\%$. Results of tolerance analysis of the voltage transfer function are illustrated on Fig. 3. Curves **a** represent the case where circuit parameters were uncorrelated. Curves **b** show the results for all capacitors and all resistors fully correlated. Curves **c** show the case in which ideal operational amplifiers were replaced by voltage controlled current sources with input resistance 40 kΩ, output resistance 10kΩ, and mutual conductance 10mS. Parameters of operational amplifiers in case **c** have normal distribution with tolerances ±10%. 95% of all circuit responses will be placed between the two curves representing each case.

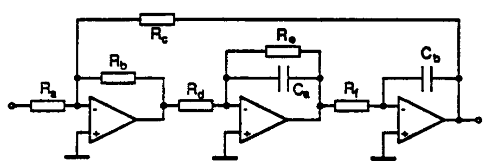

Fig. 2 Active filter circuit.

In order to compare exact and simplified analysis, case **c** was also analyzed with automatic simplification of network transfer function. Original symbolic transfer function contains 743 terms while the simplified function has only 94 terms. In both cases predicted response is almost the same and is described by the curve **c**. Time used for the symbolic analysis was reduced from 13.099s to 3.23s, and time for the tolerance analysis was reduced from 65.573s to 17.709s.

Tolerance analysis by discretization of random variables can be performed in larger networks with the help of hierarchical topological analysis [6]. As was discussed in the previous section, time needed for the tolerance analysis is proportional to the number of terms in topological formula for the transfer function. In the upward topological analysis [6] number of terms increases almost linearly with the network size.

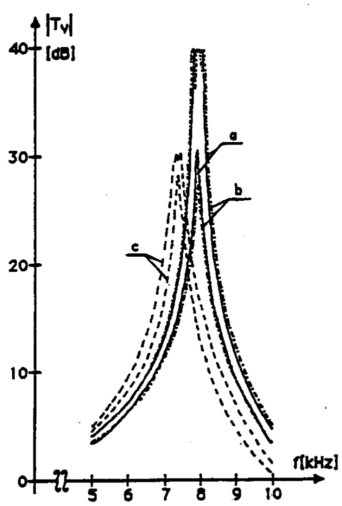

Fig. 3 Results of tolerance analysis of the voltage transfer function.

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(217) 333-4789 Further information can be obtained from Dr. W. Kenneth Jenkins.